

Fordy-Kulish model and spinor Bose-Einstein condensate

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Abstract

A three-component nonlinear Schrödinger-type model which describes spinor Bose-Einstein condensate (BEC) is considered. This model is integrable by the inverse scattering method and using Zakharov-Shabat dressing method we obtain three types of soliton solutions. The multi-component nonlinear Schrödinger type models related to symmetric spaces $\mathbf{C.I} \simeq \mathrm{Sp}(4)/\mathrm{U}(2)$ is studied.

1 Introduction

The dynamics of spinor BEC is described by a three-component Gross-Pitaevskii (GP) system of equations. In the one-dimensional approximation the GP system goes into the following nonlinear Schrödinger (MNLS) equation in (1D) x -space [1]:

$$\begin{aligned} i\partial_t \Phi_1 + \partial_x^2 \Phi_1 + 2(|\Phi_1|^2 + 2|\Phi_0|^2)\Phi_1 + 2\Phi_{-1}^* \Phi_0^2 &= 0, \\ i\partial_t \Phi_0 + \partial_x^2 \Phi_0 + 2(|\Phi_{-1}|^2 + |\Phi_0|^2 + |\Phi_1|^2)\Phi_0 + 2\Phi_0^* \Phi_1 \Phi_{-1} &= 0, \\ i\partial_t \Phi_{-1} + \partial_x^2 \Phi_{-1} + 2(|\Phi_{-1}|^2 + 2|\Phi_0|^2)\Phi_{-1} + 2\Phi_1^* \Phi_0^2 &= 0. \end{aligned} \quad (1)$$

We consider BEC of alkali atoms in the $F = 1$ hyperfine state, elongated in x direction and confined in the transverse directions y, z by purely optical means. Thus the assembly of atoms in the $F = 1$ hyperfine state can be described by a normalized spinor wave vector

$$\Phi(x, t) = (\Phi_1(x, t), \Phi_0(x, t), \Phi_{-1}(x, t))^T \quad (2)$$

whose components are labelled by the values of $m_F = 1, 0, -1$. The above model is integrable by means of inverse scattering transform method [1]. It also allows an exact description of the dynamics and interaction of bright solitons with spin degrees of freedom. Matter-wave solitons are expected to be useful in atom laser, atom interferometry and coherent atom transport. It could contribute to the realization of quantum information processing or computation, as a part of new field of atom optics. Lax pairs and geometric interpretation of the model (1) are given in [4]. Darboux transformation for this special integrable model is developed in [5]. The aim of present paper is to show that the system (1) is related to symmetric space $\mathbf{C.I} \simeq \mathrm{Sp}(4)/\mathrm{U}(2)$ (in the Cartan classification [8]) with canonical \mathbb{Z}_2 -reduction and has natural Lie algebraic interpretation. The model allows also a special class of soliton solutions. We will show that they can be obtained by a suitable modification of the generalization of the so-called “dressing method”, proposed in [9].

2 Solving the model: algebraic and analytic aspects.

The model (1) belongs to the class of multicomponent NLS equations that can be solved by the inverse scattering method [7, 6]. It is a particular case of the MNLS related to the **C.I** type symmetric space $\mathrm{Sp}(4)/\mathrm{U}(2)$ [4]. These MNLS systems allow Lax representation with the generalized Zakharov–Shabat system as the Lax operator:

$$L\psi(x, t, \lambda) \equiv i \frac{d\psi}{dx} + (Q(x, t) - \lambda J)\psi(x, t, \lambda) = 0. \quad (3)$$

where J and $Q(x, t)$ are 4×4 matrices: $J = \mathrm{diag}(1, 1, -1, -1)$ and $Q(x, t)$ is a block-off-diagonal matrix:

$$\begin{aligned} Q(x, t) &= \begin{pmatrix} 0 & \mathbf{q}(x, t) \\ \mathbf{p}(x, t) & 0 \end{pmatrix}, \quad \mathbf{q}(x, t) = \begin{pmatrix} \Phi_0(x, t) & -\Phi_1(x, t) \\ \Phi_{-1}(x, t) & -\Phi_0(x, t) \end{pmatrix}, \\ \mathbf{p}(x, t) &= \begin{pmatrix} \Phi_0^*(x, t) & \Phi_{-1}^*(x, t) \\ -\Phi_1^*(x, t) & -\Phi_0^*(x, t) \end{pmatrix}. \end{aligned} \quad (4)$$

Solving the direct and the inverse scattering problem for L uses the Jost solutions $\phi = (\phi^+, \phi^-)$ and $\psi = (\psi^-, \psi^+)$ of (3) which are defined by, see [14] and the references therein:

$$\lim_{x \rightarrow -\infty} \phi(x, t, \lambda) e^{i\lambda Jx} = \mathbb{1}, \quad \lim_{x \rightarrow \infty} \psi(x, t, \lambda) e^{i\lambda Jx} = \mathbb{1} \quad (5)$$

These definitions are compatible with the class of smooth potentials $Q(x, t)$ vanishing sufficiently rapidly at $x \rightarrow \pm\infty$. It can be shown that ϕ^+ and ψ^+ (resp. ϕ^- and ψ^-) composed by 4 rows and 2 columns are analytic in the upper (resp. lower) half plane of λ . The scattering matrix associated to (3) is defined as

$$\begin{aligned} T(t, \lambda) &= (\psi(x, t, \lambda))^{-1} \phi(x, t, \lambda) = \begin{pmatrix} a^+(t, \lambda) & -b^-(t, \lambda) \\ b^+(t, \lambda) & a^-(t, \lambda) \end{pmatrix}, \\ (T(t, \lambda))^{-1} &= \begin{pmatrix} c^-(t, \lambda) & d^-(t, \lambda) \\ -d^+(t, \lambda) & c^+(t, \lambda) \end{pmatrix}, \end{aligned} \quad (6)$$

where $a^\pm(t, \lambda)$ and $b^\pm(t, \lambda)$ are 2×2 block matrices. The blocks a^\pm , b^\pm , c^\pm and d^\pm satisfy a number of relations [11, 12]; for example

$$a^+(\lambda)c^-(\lambda) + b^-(\lambda)d^+(\lambda) = \mathbb{1}, \quad a^+(\lambda)d^-(\lambda) - b^-(\lambda)c^+(\lambda) = 0, \quad (7)$$

etc. The fundamental analytic solutions (FAS) $\chi^\pm(x, t, \lambda)$ of $L(\lambda)$ are analytic functions of λ for $\mathrm{Im} \lambda \geq 0$ and are related to the Jost solutions by:

$$\chi^\pm(x, t, \lambda) = \phi(x, t, \lambda) S_J^\pm(t, \lambda) = \psi(x, t, \lambda) T_J^\mp(t, \lambda). \quad (8)$$

Here S_J^\pm , T_J^\pm upper- and lower- block-triangular matrices:

$$\begin{aligned} S_J^+(t, \lambda) &= \begin{pmatrix} \mathbb{1} & d^-(t, \lambda) \\ 0 & c^+(t, \lambda) \end{pmatrix}, \quad S_J^-(t, \lambda) = \begin{pmatrix} c^-(t, \lambda) & 0 \\ -d^+(t, \lambda) & \mathbb{1} \end{pmatrix}, \\ T_J^+(t, \lambda) &= \begin{pmatrix} \mathbb{1} & -b^-(t, \lambda) \\ 0 & a^-(t, \lambda) \end{pmatrix}, \quad T_J^-(t, \lambda) = \begin{pmatrix} a^+(t, \lambda) & 0 \\ b^+(t, \lambda) & \mathbb{1} \end{pmatrix}, \end{aligned}$$

satisfying $T_j^\pm(t, \lambda) \hat{S}_j^\pm(t, \lambda) = T(t, \lambda)$ and can be viewed as the factors of a generalized Gauss decompositions of $T(t, \lambda)$ [13]. If $Q(x, t)$ evolves according to (1) then the scattering matrix and its elements satisfy the following linear evolution equations:

$$i \frac{db^\pm}{dt} + 2\lambda^2 b^\pm(t, \lambda) = 0, \quad i \frac{da^\pm}{dt} = 0, \quad (9)$$

so the block-matrices $a^\pm(\lambda)$ can be considered as generating functionals of the integrals of motion. The fact that all 4 matrix elements of $a^+(\lambda)$ for $\lambda \in \mathbb{C}_+$ (resp. of $a^-(\lambda)$ for $\lambda \in \mathbb{C}_-$) generate integrals of motion reflect the superintegrability of the model and are due to the degeneracy of the dispersion law of (1).

The system (1) can be written in a Hamiltonian form by introducing the Poisson brackets:

$$\{q_j(x), p_k(y)\} = 2i\delta_{kj}\delta(x-y), \quad \{q_{12}(x), p_{12}(y)\} = i\delta(x-y), \quad (10)$$

and the Hamiltonian $H = H_{\text{kin}} + H_{\text{int}}$:

$$\begin{aligned} H_{\text{kin}} &= \int_{-\infty}^{\infty} dx \left(\frac{\partial \Phi_0}{\partial x} \frac{\partial \Phi_0^*}{\partial x} + \frac{1}{2} \left(\frac{\partial \Phi_1}{\partial x} \frac{\partial \Phi_1^*}{\partial x} + \frac{\partial \Phi_{-1}}{\partial x} \frac{\partial \Phi_{-1}^*}{\partial x} \right) \right), \\ H_{\text{int}} &= - \int_{-\infty}^{\infty} dx \left((|\Phi_0|^2 + |\Phi_1|^2)^2 + (|\Phi_0|^2 + |\Phi_{-1}|^2)^2 \right) \\ &\quad - \int_{-\infty}^{\infty} dx \left(|\Phi_0 \Phi_{-1}^* + \Phi_1 \Phi_0^*|^2 \right). \end{aligned} \quad (11)$$

As mentioned above, one can use any of the matrix elements of $a^\pm(\lambda)$ as generating functional of integrals of motion of our model. Generically such integrals would have non-local densities and will not be in involution.

The classical R -matrix approach [6, 4] is an effective method to determine the generating functionals of local integrals of motion which are in involution. From it there follows that such integrals are generated by expanding $\ln m_k^\pm(\lambda)$ over the inverse powers of λ , see [13]. Here $m_k^\pm(\lambda)$ are the principal minors of $T(\lambda)$; in our case

$$\begin{aligned} m_1^+(\lambda) &= a_{11}^+(\lambda), & m_2^+(\lambda) &= \det a^+(\lambda), \\ m_1^-(\lambda) &= a_{22}^-(\lambda), & m_2^-(\lambda) &= \det a^-(\lambda). \end{aligned} \quad (12)$$

If we consider

$$\ln m_k^+(\lambda) = \sum_{s=1}^{\infty} \lambda^{-k} I_s^{(k)},$$

then one can prove that the densities of $I_s^{(k)}$ are local in $Q(x, t)$. The fact that [13]:

$$\{m_k^\pm(\lambda), m_j^\pm(\mu)\} = 0, \quad \text{for } k, j = 1, 2,$$

and for all $\lambda, \mu \in \mathbb{C}_\pm$ allow one to conclude that $\{I_s^{(k)}, I_p^{(j)}\} = 0$ for all $k, j = 1, 2$ and $s, p \geq 1$.

In particular, the Hamiltonian of our model is proportional to $I_3^{(2)}$, i.e. $H = 8iI_3^{(2)}$.

3 Soliton solutions for the spinor BEC: The $so(5)$ connection

The soliton solutions of the $\mathfrak{sp}(4)$ MNLS (1) were derived by using the dressing method [3]. They can be considered as particular cases of the soliton solutions of the generic MNLS eqs., derived through the matrix version of the Gel'fand-Levitan-Marchenko equation, see [2, 1, 3]. Here we extend further these results and combining the ideas of [3, 15] we specify three types of solitons for the model (1).

We start our analysis with the well-known isomorphism between the algebras $sp(4, \mathbb{C})$ and $so(5, \mathbb{C})$ [8]. Since the Lax representation is of pure algebraical nature it is natural to expect that our model (1) can be treated also by an equivalent Lax operator L' whose potential $Q'(x, t)$ and J' take values in $so(5)$. A consequence of the above-mentioned isomorphism is that the typical representation of $sp(4)$ used above is equivalent to the spinor representation of $so(5)$.

So we first remind some of our results in [10, 11, 12], where we have constructed the fundamental analytic solutions, the dressing factors, the soliton solutions etc. for a class of Lax operators (including L'), related to the simple Lie algebra \mathfrak{g} , in the typical representation of \mathfrak{g} . So we first have to specify (if necessary) $\mathfrak{g} \simeq so(5)$ and then reformulate the corresponding results for the spinor representation of $so(5)$.

The main goal of the dressing method is, starting from a solution $\chi_0^\pm(x, t, \lambda)$ of $L_0(\lambda)$ with potential $Q_{(0)}(x, t)$ to construct a new singular solution $\chi_1^\pm(x, t, \lambda)$ with singularities located at prescribed positions λ_1^\pm ; the reduction $\mathbf{p} = \mathbf{q}^\dagger$ used in eq. (4) ensures that $\lambda_1^- = (\lambda_1^+)^*$. The new solutions $\chi_1^\pm(x, t, \lambda)$ will correspond to a potential $Q_{(1)}(x, t)$ of $L(\lambda)$ (3) with two discrete eigenvalues λ_1^\pm . It is related to the regular one by a dressing factor $u(x, \lambda)$

$$\chi_1^\pm(x, t, \lambda) = u(x, \lambda) \chi_0^\pm(x, t, \lambda) u_-^{-1}(\lambda). \quad u_-(\lambda) = \lim_{x \rightarrow -\infty} u(x, \lambda) \quad (13)$$

Note that $u_-(\lambda)$ is a diagonal matrix. The dressing factor $u(x, \lambda)$ must satisfy the equation

$$i \frac{du}{dx} + Q_{(1)}(x)u - uQ_{(0)}(x) - \lambda[J, u(x, \lambda)] = 0, \quad (14)$$

and the normalization condition $\lim_{\lambda \rightarrow \infty} u(x, \lambda) = \mathbb{1}$. Besides $\chi_i^\pm(x, \lambda)$, $i = 0, 1$ and $u(x, \lambda)$ must belong to the corresponding Lie group $Sp(4, \mathbb{C})$; in addition $u(x, \lambda)$ by construction has poles and/or zeroes at λ_1^\pm .

The construction of $u(x, \lambda)$ is based on an appropriate ansatz specifying explicitly the form of its λ -dependence.

$$u(x, \lambda) = \mathbb{1} + (c_1(\lambda) - 1) P_1(x, t) + \left(\frac{1}{c_1(\lambda)} - 1 \right) \bar{P}_1(x, t),$$

$$c_1(\lambda) = \frac{\lambda - \lambda_1^+}{\lambda - \lambda_1^-}, \quad (15)$$

where the projectors $P_1(x, t)$ and $\bar{P}_1(x, t)$ are of rank 1 and are related by $\bar{P}_1(x) = S P_1^T(x) S^{-1}$. They must satisfy $\bar{P}_1(x, t) P_1(x, t) = P_1(x, t) \bar{P}_1(x, t) = 0$. By S we have denoted the special matrix which enters in the definition of the orthogonal algebra, i.e. $X \in so(5)$ if $X + S X^T S^{-1} = 0$. In the typical representation of $so(5)$ we have $S = \sum_{k=1}^5 (-1)^{k+1} E_{k, 6-k}$ where $(E_{ij})_{km} = \delta_{ik} \delta_{jm}$. The construction of $P_1(x, t)$ and $\bar{P}_1(x, t)$

using the ‘polarization’ vectors is outlined in [11] and we skip it. The new potential is obtained from

$$Q_{(1)}(x, t) - Q_{(0)}(x, t) = (\lambda_1^+ - \lambda_1^-)[J, P_1(x, t) - \bar{P}_1(x, t)].$$

Here we show that the λ -dependence of $u(x, \lambda)$ may depend [10] on the choice of the representation of $so(5, \mathbb{C}) \simeq sp(4, \mathbb{C})$. For $so(5)$ it was shown [10, 11, 12, 15] that there are three types of solitons:

- the first type of soliton solutions are generated by dressing factors of the form (15). For generic choice of the polarization vectors $P_1(x, t) - \bar{P}_1(x, t) \in so(5)$.
- the second type of soliton solutions are generated analogously with dressing factor (15), but due to a specific choice of the polarization vectors $P_1(x, t) - \bar{P}_1(x, t) \in so(3) \subset so(5)$.
- the third type of soliton solutions are generated again by (15) but now the corresponding projectors $P_1(x, t)$ and $\bar{P}_1(x, t)$ have rank 2.

Each of these types of soliton solutions have their counterpart relevant to our model on $sp(4)$. To the first type of soliton solutions there correspond dressing factor and potential $Q_{(1)}(x, t)$ of the form [3]:

$$\begin{aligned} \tilde{u}(x, \lambda) &= \sqrt{c_1(\lambda)}\pi_1(x) + \frac{1}{\sqrt{c_1(\lambda)}}\bar{\pi}_1(x), \\ Q_{(1)}(x, t) - Q_{(0)}(x, t) &= \frac{1}{2}[J, \pi_1(x, t) - \bar{\pi}_1(x, t)], \end{aligned} \quad (16)$$

where $\pi_1(x)$ and $\bar{\pi}_1(x)$ are *rank 2* projectors, such that

$$\bar{\pi}_1(x)\pi_1(x) = \pi_1(x), \quad \bar{\pi}_1(x) = 0, \quad \bar{\pi}_1(x) + \pi_1(x) = \mathbb{1}. \quad (17)$$

This last property ensures the non-degeneracy of $u(x, \lambda)$. Note that now the dressing factor is not a rational function of λ but for the dressed FAS $\chi(x, \lambda)$ eq. (13) we get:

$$\begin{aligned} \chi_1^\pm(x, t, \lambda) &= \left(\pi_1(x, t) + \frac{1}{c_1(\lambda)}\bar{\pi}_1(x, t) \right) \chi_0^\pm(x, t, \lambda) (\pi_1^- + c_1(\lambda)\bar{\pi}_1^-), \\ \pi_1^- &= \lim_{x \rightarrow -\infty} \pi_1(x, t), \end{aligned} \quad (18)$$

i.e., the fractional powers of $c_1(\lambda)$ disappear.

The second type of solitons with rank 2 projector $P_1(x)$ after recalculating to the spinor representation formally keeps the same form (15) with $P_1(x)$ replaced by $A_1(x)$ which has rank 1 but generically is not a projector, see [3].

The third type of solitons is similar to the second one but with additional constraints on the factor $A_1(x)$ so that $A_1(x) - \bar{A}_1(x) \in sp(2) \subset sp(4)$.

Consider the purely solitonic case when $Q_{(0)} = 0$. From now on we introduce the following notations $\lambda_1^\pm = \mu_1 \pm i\nu_1$ and

$$\begin{aligned} A &= -2i((\lambda_1^+)^2 - (\lambda_1^-)^2)t - i(\lambda_1^+ - \lambda_1^-)x, \\ B &= -2((\lambda_1^+)^2 + (\lambda_1^-)^2)t - (\lambda_1^+ + \lambda_1^-)x. \end{aligned} \quad (19)$$

Here $A(x, t)$ and $B(x, t)$ are x and t dependent real valued functions. Making use of the explicit form of the projectors $P_{\pm 1}(x)$ valid for the typical representations of \mathbf{B}_2 we obtain[11]:

$$\Phi_{(1)}(x, t) = \frac{4(\lambda_1^+ - \lambda_1^-)}{\langle m|n \rangle} (n_{0,1}m_{0,2}e^A + n_{0,\bar{2}}m_{0,\bar{1}}e^{-A}) e^{iB} \quad (20)$$

$$\Phi_{(0)}(x, t) = i \frac{2\sqrt{2}(\lambda_1^+ - \lambda_1^-)}{\langle m|n \rangle} (n_{0,1}m_{0,3}e^A - n_{0,\bar{3}}m_{0,\bar{1}}e^{-A}) e^{iB} \quad (21)$$

$$\Phi_{(-1)}(x, t) = -\frac{4(\lambda_1^+ - \lambda_1^-)}{\langle m|n \rangle} (n_{0,1}m_{0,\bar{2}}e^A + n_{0,2}m_{0,\bar{1}}e^{-A}) e^{iB} \quad (22)$$

where the denominator in the above formula is given by:

$$\begin{aligned} \langle m|n \rangle &= m_{0,1}n_{0,1}(e^{2A}) + m_{0,\bar{1}}n_{0,\bar{1}}(e^{-2A}) \\ &+ m_{0,2}n_{0,2} + m_{0,\bar{2}}n_{0,\bar{2}} + m_{0,3}n_{0,3}. \end{aligned} \quad (23)$$

and $m_{0,k}$, $n_{0,k}$ are the components of the polarization vectors.

Choosing appropriately the elements of the polarization vectors $|n_0\rangle$ and $|m_0\rangle$, one can show that the conjecture that the Zakharov-Shabat dressing procedure and the Gel'fand-Levitan Marchenko formalism lead to comparable soliton solutions is true. It is not a problem to multiply the polarization vectors $|n_0\rangle$ and $|m_0\rangle$ by an appropriate scalar and thus to adjust the two solutions. Such a multiplication easily goes through the whole scheme outlined above. The involution $Q_{(1)}^\dagger = Q_{(1)}$ that the potential of the Lax operator (3) associated with the system (1) is subject to results in the following relations between the elements of the "polarization" vectors $|n_0\rangle$ and $\langle m_0|$, namely $n_{0,k} = m_{0,k}^*$. Utilizing the above and a proper change of field components, we can relate the solution

$$\mathbf{q}(x, t) = 4\nu_1 \frac{C^\dagger e^A + \sigma_2 C^t \sigma_2 \det\{C^\dagger\} e^{-A}}{e^{2A} + W + |\det\{C\}|^2 e^{-2A}} e^{iB}, \quad (24)$$

where $W = (2|c_{12}|^2 + |c_1|^2 + |c_2|^2)$ and the "polarization" matrix can be cast into the form

$$C = \begin{pmatrix} c_{12} & c_1 \\ c_2 & -c_{12} \end{pmatrix} \quad (25)$$

In the special case when $W = 1$ and $\det\{C\} = 0$ we obtain

$$\mathbf{q}(x, t) = \frac{2\nu_1 e^{iB}}{\cosh A} C^\dagger \quad (26)$$

Thus we confirm the result obtained in[1], aquaired with the help of GLM formalism and the solution (20), derived within the generalized Zakharov-Shabat dressing procedure, provided we make sure that the extra condition on the vector $|m\rangle$:

$$-2m_{0,1}m_{0,\bar{1}} + 2m_{0,2}m_{0,\bar{2}} = (m_{0,3})^2, \quad (27)$$

and analogous one for $|n\rangle$ holds true. Setting

$$\begin{aligned} m_{0,1} &= 1, & m_{0,\bar{1}} &= -(c_{12}^*)^2 - c_1^* c_2^*, \\ m_{0,2} &= ic_1^*, & m_{0,\bar{2}} &= ic_2^*, & m_{0,3} &= m_{0,\bar{3}} = -\sqrt{2}c_{12}^* \end{aligned}$$

we establish the equivalence between the two solutions.

4 Conclusions

We have derived the soliton solutions of the three-component system of NLS type on the symmetric space $\mathrm{Sp}(4)/\mathrm{U}(2)$ which is related to spinor Bose-Einstein condensate model (with $F = 1$). Furthermore, we have described briefly the Hamiltonian properties of the model and the integrals of motion. Using the classical r -matrix approach, we showed that the integrals of motion, that belong to the principal series are in involution.

The reduction of the multi-component nonlinear Schrödinger (NLS) equations on symmetric space $\mathbf{C.I} \simeq \mathrm{Sp}(2p)/\mathrm{U}(p)$ for $p = 2$ is related to spinor model of Bose-Einstein condensate. Other interesting reductions of MNLS type equations were reported in [11] and a systematic study of the problem is on the way. Recently the authors of [16] develop a perturbation theory for bright solitons of the $F = 1$ integrable spinor BEC model. Both rank-one and rank-two soliton solutions are obtained using Riemann-Hilbert method and are compared with known results.

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